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ON AN ALGEBRAIC CHARACTERIZATION OF HIERARCHIC SYSTEMS

In the paper it is shown that the acting (the operating) of left regular bands on some sets determines hierarchic structures particularly Paun's systems and characterize some properties of these structures.

hierarchic systems, higher hypergraphs, globularity, data bases

Introduction

Graphs can be defined as unary algebras of the form $G = (X,\ s,\ t)$ with $s,\ t : X \to X$ being functions satisfying the conditions:

$$s \circ s = s \circ t = s \quad \&; \qquad t \circ s = t \circ t = t.$$

This definition leads to a presentation of Petri nets as „relational algebras" of the form (X, entry, exit) satisfying similar conditions [7] and may be generalized to Petri nets seen as „relational graphs over independent systems" in [8]. In this paper it is shown that the method used in [7] can also be used to a characterization of hierarchical systems and data bases with a kind of refinement of attributes.

1. Preliminaries

1.1. ***Notation.*** *In the* paper the standard mathematical notation is used. By a fix point of a function $f : X \to X$ we mean an element $x \in X$ satisfying the condition $f(x) = x$. fix(f) denotes the set of all fix points of f. The set of all functions from a set A to a set B will be denoted by $[A \to B]$ or B^A. The composition of functions $f : A \to B$ and $g : B \to C$ will be denoted by fg or $f \circ g$. So we write $fg(a) = g(f(a))$ for $a \in A$. If $A = B$ than we write simple Funct(A) instead of $[A \to A]$. For any unexplained notions and notation the reader is referred to [10].

1.2. Graphs and hypergraphs. The "standard" presentation of directed (multi)graphs as set valued functors defined on the category

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \arrow[loop right, "id_V"] \end{tikzcd}$$

may be generalized to the so called **higher graphs** being set valued functors defined on the category

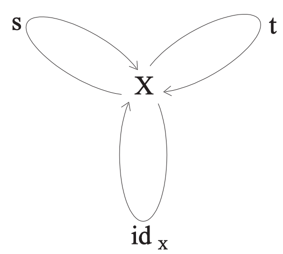
$$X_0 \xleftarrow{\{s_0, t_0\}} X_1 \leftarrow \ldots \xleftarrow{\{s_{n-1}, t_{n=-1}\}} X_n \qquad (1)$$

(in the above picture the identitity morphisms have been omitted). Such structures are closely related to the so called n-categories (see e.g. the web page on n-categories [12]). **Higher hypergraphs** may be seen as as set valued functors defined on the category

$$X_0 \xleftarrow{F_0} X_1 \xleftarrow{F_1} X_2 \longleftarrow \ldots \xleftarrow{F_{n-1}} X_n \quad (2)$$

with $F_0, F_1, \ldots, F_{n\text{-}1}$ being **arbitrary** sets of morphisms.

One sorted graphs may be seen as set valued functors defined on the category whose underlying graph is given in the following figure



and the composition of morphisms satisfies the condition

$$t \circ\ s = t \circ t = s \ \text{ and } \ t \circ s = t \circ t = t.$$

Such a graph is an algebra of the form (X, S) where S is a three element left zero monoid of functions defined on the set X (acting on the set X). The vertices of such graphs are the common fix-points of the operations s and t. Generalizing this notion we can define directed hypergraphs as unary algebras of the form $H = (X, (\delta_s)_{s\in S})$ satisfying the condition: for any $x \in X$ and any $s, s' \in S$ we have $\delta_s(\delta_{s'}(x)) = \delta_{s'}(x)$. The elements of the set[1]

$$V(H) = \{x \in X\colon \delta_s(x) = x\}$$

are called **Vertices** of H and those from the set $A(H) = X\setminus V(H)$ **hyperarrows** of H. The set I is called the type of H[2]. Homomorphisms of hypergraphs are defined as the standard homomorphisms of algebras that means by a homomorphism from a hypergraph $H = (X, (\delta_s)_{s\in S})$ into a hypergraph $H' = (X', (\delta')_{s\in S})$ we understand any function $f: X\to X'$ satisfying the condition

$$\delta_s'(f(x)) = f(\delta_s(x))$$

for any $x\in X$ and $s\in S$. A hypergraph $H = (X, (s_s)_{s\in S})$ may be now seen as an action of a left zero semigroup, say $S = (S, \circ)$, on the set X. Now one can define **higher hypergraphs** as sequences of the form

$$X \xleftarrow{F_0} X \xleftarrow{F_1} X \longleftarrow \ldots \xleftarrow{F_{n-1}} X$$

with $F_0, F_1, \ldots, F_{n-1}$ being arbitrary sets of morphisms satisfying the condition $\delta_s \circ \delta_\tau = \delta_s$ for any $I \leq n-1$ and any $\delta_s, \delta_t \in F_i$. So, a higher hypergraph may be seen as a sequence of left zero semigroups acting on the same set X. For any category $\mathcal{C}$, a **$\mathcal{C}$-hypergraph** is a functors defined on $\mathcal{C}$. The object part of such a functor, let us call it $\Lambda\colon \mathcal{C}\to$ set is a constant function with a value being a fixed set, say X and for any objects $a, b \in$ Objects($\mathcal{C}$) the set Mor($\Lambda(a)$, $\Lambda(b)$) being a left zero semigroup. If we treate $\mathcal{C}$ **as a partial semigroup** then $\mathcal{C}$-hypergraphs are simply some special homomorphisms of semigroups. It can be shown that the codomains of these homomorphisms are left regular semigroups.

[1] Let us note that the condition $\delta_s(x)=x$ guarantees that for any $s'\in S$ we also have $\delta_{s'}(x)=x$ which legalize the notation.

[2] In the case of general algebras it is better to define the type of an algebra as a **sequence of operations** (if the set of these operations is finite), a sequence of operators (operation symbols) or as an ordinal number (see e.g. [5]). In this case the type may be understood as a set, because all operations have the same arity and the index-function

$$I \ni i \quad s\in \text{operations\_of\_the\_hypergraph}$$

suffices for the identification of operations in the considered hypergraphs.

1.3. Bands. By a **band** it is meant any idempotent semigroup i.e. a semigroup in which every element satisfies the equality $xx = x$ that means every element of S is idempotent. Let us note the following simple fact.

Proposition. A function $f\colon X\to X$ is idempotent iff $\mathrm{fix}(f) = f(X)$ iff $f\subseteq \ker(f)$.

It is well known that every semigroup can be represented by a semigroup of functions of the form $X\to X$ for *a* given set X[3]. Let X be a given set. In what follows we consider bands of functions that means semigroups of the form $\mathrm{B}=(S,\circ)$ with $S\subseteq(\mathrm{Funct}(X)$ in which any element (function) f satisfies the condition $\mathrm{fix}(f) = f(X)$ (equivalently $f\subseteq\ker(f)$.

Proofs of the following propositions consist if simply calculations.

Proposition. For any band of functions $\mathrm{B} = (S, \circ)$ and $f,g\in S$ it holds $f\circ g=f$ iff $\mathrm{fix}(f)\subseteq\mathrm{fix}(g)$.

Corollary. $\mathrm{fix}(f)=\mathrm{fix}(g)$ iff $f\circ g=f$ & $g\circ f=g$.

The relation $\prec_{\mathrm{fix}}\subseteq\mathrm{Funct}(X)^2$ given by the condition $f \prec_{\mathrm{fix}} g \Leftrightarrow \mathrm{fix}(f)=\mathrm{fix}(g)$ is of course an equivalence in the set Funct(X).

The equivalences classes of this relation will be called **levels** of B. Of course every such a level is uniquely determined by the corresponding set of fix-points. In what follows the level determined by a set $A\subseteq X$ will be denoted by level(A). The level determined by a set $A\subseteq X$ will also be called the **level of a function** $f\colon X\to X$ for any function f with $\mathrm{fix}(f)=A$. Let us recall that it is a set of functions level(A)$\subseteq$Funct(X). Let X be a fixed set and $\mathrm{B}=(S, \circ)$ with $S\subseteq\mathrm{Funct}(X)$ a band of functions.

Proposition. For every subset A of the set X the level of A is a left zero subsemigroup of B.

Proposition. Every subset of (the carrier of) a level is a subsemigroup of this level.

[3] Perhaps the best well known such a representation is the representation of a semigroup as a semigroup of translations, that means mappings of the form $\Lambda_a\colon X\to X$ with $\Lambda_a(x)=ax$ for a semigroup $S=(X, \cdot)$ and a fixed element $a\in X$.

2. Representation theorem

It is well known (see e.g. [14], p.20, theorem II.1.6) that any left regular band $B = (X, \circ)$ may be represented as the (unions of) a semilattices of left-zero semigroups. From the definition of higher hypergraph we obtain.

Proposition. *A* pair (X, F) is a higher hypergraph iff $(F, \circ)$ is a left regular semigroup of functions $F \subseteq [X \to X]$ and $F/_{\text{commutativity}}$ [4] is a chain. The „length" of this chain is the length of the higher hypergraph.

3. Globularity

By a **globular set** it is meant any higher graph

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2 \leftleftarrows \ldots \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} X_n$$

satisfying the condition: for any $i \leq n-1$

$$s_i \circ s_{i-1} = t_i \circ s_{i-1} \;\&\; t_i \circ t_{i-1} = s_i \circ t_{i-1}.$$

This condition originates from the request that such a globular set should play the role of the underlying (n-) graph of an n-category, i.e. if the graphs

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1,\; X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2,\; \ldots,\; X_{n-1} \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} X_n$$

are the underlying graphs of some categories C_1, C_2, ..., C_{n-1} then the globular set may be seen as the „underlying n-graph" of the n-category with „levels" C_1, C_2, ..., C_{n-1}. For more details the reader is referred to [1] and [2].

Globularity has an interesting interpretation in data bases. Firstly, let us note that any table may be easily seen as a hypergraph with hyperarrows being (named) rows (records) of the table and incidence functions being single projections onto attributes (vertices are values of attributes). We can say that a table is a hypergraph with an additional partition of vertices into sets corresponding to the columns of the table. In this sense one can call tables n-partite hypergraphs. We consider tables as Pawlak's information systems (see [13]), that means quadruples of the form[5]

$$H=(\text{Objects, Attributes, Values; Information}),$$

where Objects, Attributes and Values are sets and information is a function of the form

$$\text{Information: Objects} \times \text{Attributes} \to \text{Values}.$$

Let us consider a table

attributes / objects	A_1	A_2
O_1	a_{11}	a_{12}
O_2	a_{21}	a_{22}
...	...	...

It may be seen as the information system

$$H_0 = (\text{Objects}_0, \text{Attributes}_0, \text{Values}_0; \text{Information}_0),$$

where

$$\text{Information}_0\text{: Objects}_0 \times \text{Attributes}_0 \to \text{Values}_0$$

and

$$\text{Values}_0 = \bigcup_{A \in Attributes_0} values_0(A)$$

with $\text{Values}_0(A)$ being a set (of the values of the attribute A). Such a table gives a description of (elements of) the set Objects_0 characterizing its elements by the rows of the table named in this context records. One assumes that (the values of) attributes are atomic and there exists no need to refine (decompose) them into „better known parts". However this assumption may sometimes fail. In such a case we have to see (the values of) attributes as objects of another, in a sense „detailed", information system. So, we obtain a new table

New attributes / values of the attributes A_i	B_1	B_2
a_{11}	b_{11}	b_{12}
a_{11}	b_{21}	b_{22}
a_{21}	b_{31}	b_{32}
a_{22}	b_{41}	b_{42}
...	...	...

which leads to a new information system. Now, one may merge both systems and use it "depending on the user". If there is a need for the information on the level determined by the first table (let us call it the zero-level table) then we use the function Information_0 only and if

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[4] Commutativity is the relation generated in F by the equality fg = gf.

[5] This is an nonessential modification of Pawlak's definition of an information system.

a “deeper” information is needed we offer a set of tables. In this set some tables may refine the other ones. If this refinement relation is a chain then we offer the user an n-hypergraph. The levels of this hypergraph are exactly the information systems H_0, H_1, ... and the sets F_0, F_1, ... consist of “refining functions” assigning to values of attributes some records of a new data-base being their refinements

$$\ldots \xleftarrow{F_1} H_1 \xleftarrow{F_0} H_0 \,.$$

The assumption that such a „refinement sequence” fit to any user is very optimistic. Normally any person has his own way of understanding of a notion (a value of an attribute). An important property of the “goodness” of such a refinement is the property that for any two persons, say P_1 and P_2 there exists a common refinement of a given table which may be understood by both P_1 and P_2. If we assume already such a preexisting refinement then we obtain the statement

The refinement structure of a date base is a semilattice[6].

The above interpretation of hypergraphs is explicit in this sense that any hypergraph may be seen as a (generalized) data base (table). The dual reasoning (which we omit here) leads to the notion of a database with the operation of generalization.

Now, what are the elements of the semilattice mentioned above? They are of course the tables that means the hypergraphs being the corresponding Pawlak's system.

What does it mean that the refinement system of a data base (knowledge base) is globular? Let us consider a record α in a data base A with attributes A_1, A_2, A_3, A_4. The corresponding incidence functions are f_1, f_2, f_3, f_4

α :

A_1	A_2	A_3	A_4
$f_1(\alpha)$	$f_2(\alpha)$	$f_2(\alpha)$	$f_2(\alpha)$

Refining a area, say for example A_1, by two functions g_1 and g_2 we obtain different, more precise with different names, records say β_1 and β_2

$\beta_1 = g_1(\alpha)$:

B_1	B_2	B_3
$h_1(\beta_1)$	$h_2(\beta_1)$	$h_3(\beta_1)$

$\beta_2 = g_2(\alpha)$:

B_1	B_2	B_3
$h_1(\beta_2)$	$h_2(\beta_2)$	$h_3(\beta_2)$

in a data base (table) B with attributes and

$$h_1, h_2, h_3\text{: Objects}(B) \times \{B_1, B_2, B_3\} \to \text{Values.}$$

The globularity of the refinement system says that

$$h_i(g_1(\alpha)) = h_i(g_2(\alpha))$$

for any $i \le 3$. In other words the records β_1 and β_2 may differ only in names, one can say they are isomorphic. Such a property may be important when considering knowledge bases in various expert systems where the problems connected with the identification of objects are of great importance.

4. Hierarchical systems

The above considerations can be easily transformed onto endomorphisms semigroup of any object. Let us consider such a semigroup of a semilattice with the unit and zero element, that means a triple of the form[7]

$$\text{IS} = (X, \cup, 1, 0)$$

with $1,0 \in X$ being distinguished elements of X and $\cup$: $X \times X \to X$ being an idempotent, assotiative and commutative binary operation in X satisfying the conditions

$$X \cup 1 = 1 \cup X = X \;\&\; X \cup 0 = 0 \cup X = 0.$$

In other words we consider (hyper)graphs over independence systems. In [8] it has been shown that graphs over such independence systems can be seen as Petri nets. Such a presentation of a Petri net is a graph over an independence system IS = $(X, \cup, 1, 0)$. The Vertices of and the Arrows of this graph are (correspond to)

[6] This statement may be a bit too powerful. A table with “infinite rows” is useless. Similar, if an attribute is not defined on an object (e.g. the attribute “weight” of a song) then it may be very artifical to see such a table as a hypergraph. However we can simply consider finite semigroups of operations and introduce a special “bottom element”.

[7] Such a monoid can be understand as a model of an independence system (see [8]), more precisely as the image of such a system by a lattice homomorphism transforming all dependent objects (usually sets) onto the zero of IS. IS is an abbreviation of **I**ndependence **S**ystem.

situations and process in this net. The corresponding endomorphism semigroup of IS has two elements

$$\text{entry, exit}: X \to X \qquad (\text{entry} \neq 1 \neq \text{exit})$$

assigning to any process α of X its beginning (entry(α)) and end (exit(α)) situation. The fact that entry and exit are respected by the operation „$\cup$"may be understand as a kind of safety of the process algebra of this net (see e.g. [11] or [7]). Replacing the endomorphism semigroup ({entry, exit}, $\circ$) by an arbitrary left zero endomorphism band of $\mathrm{B}_{\mathrm{IS}} = (A, \circ)$ of the independence system IS we obtain the notion of generalized Petri nets (see [6]). Now hierarchic „generalized Petri nets" may be characterized by left regular bands of endomorphisms of such independence systems that means higher hypergraph. The idea is very simple. Let (X, F) be a higher hypergraph over an idependence system IS. Let

$$\mathrm{IS}^1 = (X^1, \cup, 1, 0),\ \mathrm{IS}^2 = (X^2, \cup, 1, 0),\ \ldots,$$
$$\mathrm{IS}^n = (X, \cup, 1, 0)$$

be a sequence of subsystems of IS = (X, $\cup$,1, 0). If every level F_{X^i} of F determined by the set X^i ($i \leq n$) has exactly two elements, say f and g then the pair N_A= (X, f, g) is of course a Petri net. The places of this net are processes in the „next" nets – the levels determined by the sets of the form $A \setminus \{a\}$ ($a \in A$). More precisely such a „next" net has exactly the same places as $N_{A\setminus\{a\}}$ except the element a which becomes a transition in the net $N_{A\setminus\{a\}}$. In the case of generalized nets is the reasoning similar.

Concluding remarks

Higher graphs are considered usually in connection with the higher categories mentioned in the Introduction. They have been introduced in 60-thies by Benabou [3]. Higher categories are intensively examined not only as interesting mathematical objects but also as a description tool for many objects considered e.g. in physics and computer sciences (see e.g. [4]). An excellent surrey of the theory and application of higher categories can be found on the web page [12]. Connections between left regular bands and higher graphs have been firstly described probably by [9] who presented them in an interesting philosophical context. Paun's systems may be characterized in terms of left regular bands as higher hypergraphs or dual by right regular bands as systems hierarchies) of partitions. Paun's systems may be models of many objects. Hieragchic systems and data bases are some of them.

Hierarchical organized systems of areas (e.g. zones in the air or on the earth) may be seen as Paun's systems (with additional adjancy-relation). So such systems may be described as some special left regular bands. In such a description we drop many information on regions; theirs location, their area, their shape and many other. However this abstraction allows to consider the essence of flying control; the relations between its elements. And this is exactly the point which is interestiong for a flight controller. So the possibile application of the presented formalism is a model which is simpler than models using many dimensional object as representation of zones.

One of the way of understanding the coherence of a data base is that any aggregation of such a base should be coherent.

If one considers data bases as some sets of records the inconsistency involves concerns simply a single record and the data base may be "improved" by the deletion of this record (records if there are more such "failed" records). However there do exist "data bases"[8] in which some relations between records are of great importance. In such cases the improvement of the whole system may demand (request) the removal of a subsystem generated by the failed records. The identification ("computation") of such a failed subsystem is often a very difficult problem. The globularity of an n-hypergraph corresponding to the semilattice of possible refinements of a data base offers quite a formal (practically syntactic) method of verifying the coherence of

[8] I write here „data bases" in order to be coherent with the terminology of Pawlak's information systems and the language of the theory of the (classical relational) data bases. Systems "containig" relations between records are more similar to the so called "data mines" using by expert systems.

data bases. If the emilatice of **refinements of** a data base is not globular, then the data base is not coherent. This criterion may, of course, not be used also as a positive criterion of the coherence of a criterion. However globularity seems to be a good candidate for the first examination of the set (in fact the semilattice) of the refinements of a data base.

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