UDC 681.513.7

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ON DIRECT AND INVERSE PROBLEMS FOR FUZZY EQUATION SYSTEMS WITH TOLERANCES

We investigate the resolution of fuzzy (relational) equation systems with tolerances which are a certain extension of fuzzy equations considered f.i. in [3-5]. The extension of the concept of Higashi and Klir [3] enables us to describe the set of solutions to our problem (for given tolerances) by means of posets. In a second part we investigate an inverse problem: Given upper (lower) tolerances how to determine lower (upper) tolerances such that the arising problem becomes consistent? Numerical examples are given.

fuzzy equation systems, fuzzy tolerances, posets, extremal elements, inverse problems.

1. Introduction

1.1. Direct problems

Let us consider

$$\underline{B_i} \subseteq R_i \circ X \subseteq \overline{B_i}\text{ , i=1, …,N.} \tag{1}$$

$\underline{B_i}, \overline{B_i}$ are given fuzzy sets over a basic space γ. The R_i are fuzzy relations over χ×γ where χ is another basic space. We are seeking for a fuzzy set X (over χ) fulfilling (1). Furthermore we suppose that there are given fuzzy sets Bi over γ with $\underline{B_i} \subseteq B_i \subseteq \overline{B_i}$ '∘' means a convenient rule of composition. Hereforth we denote this problem by (P_D). We may interpret (P_D) as a generalization of the following problem: Find X fulfilling

$$R_i \circ \mathrm{X} = B_i\text{, i=1,..., N.} \tag{2}$$

We denote this problem by (P_{DE}). It originates among others from the fuzzification of classical nonlinear equation systems. The functions occurring there have been replaced by fuzzy relations R_i, the right-hand sides by B_i and the unknowns by X. The situation that we investigate (P_D) instead of (P_{DE}) is partially reasoned by the well-known fact that (P_{DE}) is not always solvable (cf. [4] and [5]). There are several possiblities to generalize (P_{DE}) and to define a new conception of solvability, respectively. We refer to Pedrycz [4] where a solution of (P_{DE}) is determined in a numerical environment, and to Gottwald [1] where a solvability index is used for characterization of the solution set. The above stated problem (P_D) has two advantages in our opinion: On one hand, the demand to the user to determine the tolerances $\underline{B_i}, \overline{B_i}$ seems to be fairly transparent – he may predict where tolerances (and in which quantities) are acceptable for him. On the other hand, we are able to describe the structure of the solution set of (P_D) completely in the case of finite χ, γ whereby the corresponding algorithms can be realized in an easy manner. However, sometimes the user may have some troubles to assign precise numerical values to the fuzzy sets. Gottwald and Pedrycz [2] proposed an interesting way for obtaining lower and upper tolerances using fuzzy sets which express the quality of fulfillment of (2) and which are built up applying Gottwald's solvability index. Another possibility consists in modelling the tolerances by fuzzy sets of Type 2 which has been considered by the authors in [6].

The limitation to finite basic spaces does not seem to be too restrictive, since many practical problems can be grasped in this way. However, some of the achieved results are also valid for compact χ, γ (e.g. in Euclidean space), when the corresponding membership-functions are upper semi-continuous. Nevertheless, the numerical handling of the rule of composition (to mention only one item) will be rather complicated so that we restrict ourselves to finite sets.

1.2. Inverse problems

(P_I^1): Let B_i, R_i be given and upper tolerances $\overline{B_i}$. How to determine the lower tolerances $\underline{B_i}$ such that the problem (P_D) becomes consistent (there is at least one solution X)?

(P_I^2): See (P_I^1), but this time the lower tolerances $\underline{B_i}$ are given.

Both problems are of interest from the standpoint of application: On one hand, we are able to evaluate the 'amount of consistency' if we are able to solve (P_I^1) and (P_I^2) in a certain 'optimal' manner (the greater the necessary tolerances the more contradictions we have in our problem). On the other hand, it may be interesting to know much about the tolerances given by practical demands, especially the difference to those necessary for consistency.

For the solution of (P_D), (P_I^1) and (P_I^2) we particularly use a method described by Higashi and Klir in [2] for the solution of (P_{DE}) in the case N = 1. In Section 2 we deal with a generalization of this method to our problems. Section 3 contains some more basic definitions from fuzzy set theory. The next two sections are devoted to the solution of the problems under consideration. Numerical examples are given to enlight the situation. In the 6th section we summarize and give an outlook as well.

2. Some facts from the theory of partialiy ordered sets (posets)

In the sequel we assume P to be a poset with partial order '$\leq$'.

Definition 2.1. Let $\breve{p} \in$P. We call it a minimal element for P iff for arbitrary p $\in$ P we have p = $\breve{p}$, whenever p $\leq \breve{p}$.

Definition 2.2. We call $\hat{p} \in$ P the greatest element of P iff for all p $\in$ P we have p$\leq \hat{p}$.

Henceforth we denote the set of all minimal elements of P by P_0. Then we can state the following lemmata:

Lemma 2.1. Let P be finite. Then we haue $P_0 \neq \emptyset$ and for all p $\in$ P we can find $\breve{p} \in P_0$ with $\breve{p} \leq$ p.

Lemma 2.2 (see [3]). Let P′ $\subseteq$ P. If there is for all p $\in$ P a p′$\in$ P′ with p′$\leq$p then we have $P_0 = P_0'$.

We observe that the inclusion $P_0 \subseteq P'$ follows from the above assumptions and it should not enter into the formulation of the lemma (as stated in [3]).

Now we define a point-to-set mapping F as follows:

1. F : P→ρ (P) with ρ (P) the power set of p.
2. Let p $\in$ P and q $\in$F(p). Then q$\leq$ p holds.
3. Let $p_1, p_2 \in P$, and $p_1 \leq p_2$. Then we have $F(p_1) \subseteq F(p_2)$.
4. For all p $\in$ P the set F(p) has at least one minimal element. The following theorems have been proved in [3] for (P_{DE}) and N = 1. Here we shall give generalizations useful for our objective.

Theorem 2.1. We have p $\in P_0$ iff F(p) = {p}.

Proof. (a) Let p $\in$ P_0. From property 2 of F we have from q$\in$ F(p) that q$\leq$ p, hence q = p and F(p) = {p}.

(b) Suppose F(p) = {p}. Moreover let p′$\leq$p. From property 3 we obtain F(p′) $\subseteq$ F(p), i.e. F(p′) = {p}. From q $\leq$p′$\leq p$ with q $\in$ F(p′) we get p′ =p, hence $p \in P_0$.

Theorem 2.2. Let p $\in$ P. Then there exists p′ $\in$ P_0 with p′ $\leq$p when there exist a point-to-set mapping F for poset P.

Proof. Let p $\in$P. We take p′ $\in$ F_0(p) the existence of which is ensured by property 4. From property 2 we have p′$\leq$p. Now assume p′$\notin P_0$. From Theorem 2.1 then there is q′ $\neq$p′ with q′ $\in$ F(p′). Because of property 3 we get q′ $\in$ F(p). Hence p′$\notin F_0$(p). This contradiction ends the proof.

Theorem 2.3. Suppose $\hat{p}$, to be the greatest element
of P. Then the following inclusions hold:

$P_0 \subseteq F(\hat{p}) \subseteq P$.

Proof. We must only show that $P_0 \subseteq F(\hat{p})$. Let $p \in P_0$. Then of course $p \le \hat{p}$ and therefore $F(p) \subseteq F(\hat{p})$. From Theorem 2.1 we get p = F(p), that is $p \in F(\hat{p})$.

Theorem 2.4. Let $\hat{p} \in P$ be the greatest element of P. Then we have $F_0(\hat{p}) = P_0$.

Proof. Put $P' = F(\hat{p})$. From Theorem 2.2 for all p $\in$ P we can find $\breve{p} \in P_0$ with $\breve{p} \le p$. From Theorem 2.3 we have $P' \ni \breve{p}$. Application of Lemma 2.2 concludes the proof.

The last statement enlights the usefulness of the mapping F: If $\hat{p}$ and $F_0(\hat{p})$ are easy to obtain (in comparison with a straightforward determination of P_0) the above method may be advantageous.

3. Basic definitions from fuzzy set theory

Let $\chi = \{\chi_1,...,\chi_m\}$, $\gamma = \{\eta_1,...,\eta_n\}$. By $\varphi(\chi), \varphi(\gamma), \varphi(\chi \times \gamma)$, we denote the sets of all fuzzy sets over $\chi, \gamma, \chi \times \gamma$ (Cartesian product). Let X $\in \varphi(\chi)$. By μ_χ we denote the corresponding membership function. For sake of lucidity we will also use the following abbreviations:

$\mu_x(\chi_i)=x_i$, $\mu_y(\chi_j)=y_i$, $\mu_{Rk}=(\chi_i, \eta_i)=r^k_{ij}$

Definition 3.1 (Rule of composition).
Let h : [0, l] x [0, l]→[0, l] be a continuous t-norm (see [l] and [4]). Furthermore let X $\in \varphi(\chi)$, R $\in \varphi(\chi \times \gamma)$. Then we define Y = R$\circ$X$\in \varphi(\gamma)$ by

$y_i = \max_{1 \le i \le m} h(x_i, r_{ij})$, j = 1, . . . , m.

Definition 3.2. Let X_1, X_2, be two fuzzy sets over the same basic space. We say that $X_1 \subseteq X_2$ iff $\mu_{x1} \le \mu_{x2}$ for all elements of the basic space.

Definition 3.3 (a-operation). Let a, b $\in$ [0, 11. Then we put

$$a\alpha b = \begin{cases} 1 & for\ a \le b, \\ \max\limits_{z \in h^{-1}(a,b)} z & for\ a > b, \end{cases}$$

with h^{-1} (a, b) = {z $\in$ [0, l] : h(a, z) = b}.

Sanchez was the first who used a-operations (among others) to resolve fuzzy relation equations (see also [4]).

Detinition 3.4. Let Y$\in \varphi(\gamma)$, R$\in \varphi(\chi \times \gamma)$.

Then we define X = R α Y by $x_i = \min_{1 \le j \le n} r_{ij} \alpha y_j$, i = 1, . . . , m.

By the help of Definitions 3.1. and 3.2. we are now able to give an exact mathematical formulation of our problems (P_D), (P^1_I) and (P^2_I).

4. The direct problem (P_D)

Now we will investigate (P_D). By ψ we denote the set of all solutions of (P_D). Further we suppose $\varphi(\chi), \varphi(\gamma)$ and $\varphi(\chi \times \gamma)$ to be partially ordered by inclusion according to Definition 3.2. Next we will give a simple criterion for ψ ≠ 0. As a first step we state:

Theorem 4.1. Put $X_i^{\max} = R_i \alpha \overline{B}_i$, i = 1, . . . , N. Then

$$X^{\max} = \bigwedge_{i=1}^{N} X_i^{\max} \tag{3}$$

represents the greatest fuzzy set fulfilling

$$R_i \circ X \subseteq \overline{B}_i,\ i=1,...,N \tag{4}$$

(here '$\wedge$' denotes the min-operator being applied componentwise). We can state the following important result:

Theorem 4.2. We have ψ ≠ 0 iff $X^{\max} \in$ ψ. In this case $X^{\max}$ is the greatest element of ψ. For the proofs we refer to [4] and [5]. We will deal now with the construction of a mapping F as defined in Section 2.

For simplification we set $\mu_{Bi}(\gamma\kappa) = \underline{b^i_k}$ and similarly for $\overline{B_i}$. Let X $\in \varphi(\chi)$. We define

$$I^i_k(X) = \{j \in \{1,...,m\} : h(x_j, r^i_{jk}) \ge \underline{b^i_k}$$
$$I^i(X) = \mathop{\chi}_{k=1}^{n} I^i_k(X), n = 1,...,N \tag{5}$$

(our notation is connected with that in [2]). We choose $f^i \in I^i$ (X) and construct the following mapping:

[f^i (X)]: $\psi \to \varphi(\chi)$ with

$$\left[f^i(X)\right]_J = \begin{cases} \max\limits_{K\in J_{f^i}(j)} \ \min\limits_{z\in h^{-1}(r^i_j k, \underline{b}^i_k)} z & for j_{f^i}(j) \neq \emptyset, \\ 0 & for J_{f^i}(j) \neq \emptyset, \end{cases} \quad (6)$$

with j running from 1 to m. The index set $J_{f^i}(j)$ is defined as follows: Let $f^i = (f^i\ (l), \ldots, f^i\ (n))$. Then $J_{f^i}(j)$ (j) = {I ∈ (1, . . .) n}: F^i (l) =j}. Finally we set

$$f = \mathop{X}_{i=1}^{N} f^i \in I(X)$$

with $I = X_{i=1}^{N} I^i(X)$ and define a mapping [f(X)] : $\psi \to (\varphi \times \chi)$ by

$$[f(X)]_i = \max_{1\le i\le N}\left[f^i(X)\right]_j. \quad (7)$$

Now we can give the desired definition of F:

$$F(X) = \{[f(X) : f \in I(X)\ \}] \quad (8)$$

Theorem 4.3. The following statements are true:

(a) For X ∈ ψ we have F(X) ⊆ ψ.

(b) Let f ∈ I(X). Then [f(X)] ⊆ X.

(c) Let X, Z ∈ ψ with X ∈ Z. If [f(X)] ∈ F(X) then we obtain [f(Z)] ∈ φ (Z) and [f (X) = [f (Z)].

Proof. (b) Let X ∈ ψ and suppose [f(X)] ⊄ X. Then there is a j* ∈ (1, . . . ,m} with $[f(X)]_{j^\bullet} > x_{j^\bullet}$.

Hence we can find an i*∈ (1,…,n} with $[f^{i\bullet}(X)]_{j\bullet} > x_{j\bullet}$ .Consequently $J_{f^{i\cdot}}$ (j*) ≠ 0 and there is a k*∈ $j_{f^i}(j^*)$ such that inf z >x_i with z ∈ $h^{-1}(r^{i\bullet}_{j^\bullet k^\bullet}, \underline{b}^{i\bullet}_{k^\bullet})$. But this implies $h(x_{j^\bullet}, r^{i\bullet}_{j^\bullet k^\bullet}) < \underline{b}^{i\bullet}_{k^\bullet}$ which is a contradiction to k* ∈ $j_{f^{i\bullet}}$ (j*).

a) It remains to show that $R_i \circ Z \supseteq \underline{B_i}$, i = 1, . . .,N, with Z = [f(X)]. Assume that this inclusion is not valid. Then there are k*∈ { 1, . . . , n}, i*∈ (1, . . . , N} with $(R_{i^\bullet} \circ Z)_{k^\bullet} < \underline{b}^{i\bullet}_{k^\bullet}$ from which we immediately get $h(r^{i\bullet}_{j^\bullet k^\bullet}, z_j) < \underline{b}^{I\bullet}_{k^\bullet}$ j= 1 …, m. From X ∈ ψ we find a $\bar{j} \in$ { 1, . . . , m} with $h(r^{i\bullet}_{j^\bullet k^\bullet}, x_{\bar{j}}) \ge \underline{b}^{I\bullet}_{k^\bullet}$, that is, $j_{fi^\bullet}(\bar{j}) \neq \emptyset$. But then we get in particular $h(r^{i\bullet}_{j^\bullet k^\bullet}, \left[f^{i^\bullet}(X)\right]_{\bar{j}}) < \underline{b}^{I\bullet}_{k^\bullet}$. From this and (6) we obtain $h(r^{i\bullet}_{j^\bullet k^\bullet},\ \inf z) < \underline{b}^{I\bullet}_{k^\bullet}$ with z ∈ $h^{-1}(r^{i\bullet}_{j^\bullet k^\bullet}, \underline{b}^{I\bullet}_{k^\bullet})$. But this is a contradiction to the definition of h^{-1} and the continuity of h.

(c) Suppose X, Z ∈ ψ and X⊆ Z. We choose i ∈ {1, . . . , N} and consider $f^i \in I^i(X)$. From $x_j \le z_j$ for all j ∈ {1, . . . , m} we have $f^i \in I^i$ (Z) and therefore $\left[f^i(X)\right] = \left[f^i(Z)\right]$ and $\left[f(X)\right] = \left[f(Z)\right]$.

This theorem shows that the above constructed mapping F satisfies the conditions of the second section. The fourth condition (existence of minimal elements of F(X)) is trivially fulfilled, since F(X) is a finite finite set and therefore Lemma 2.1 holds. As a consequence we get:

Theorem 4.4. The sets ψ and $F(X^{\max})$ have the same minimal elements, i.e. $\psi = F_0(X^{\max})$.

With respect to the structure of ψ we formulate:

Theorem 4.5 (Representation Theorem). The following equality is valid:

$$\psi = \bigcup_{\breve{X}\in\upsilon_0} \left[\breve{X}, X^{\max}\right],$$

where $\left[\breve{X}, X^{\max}\right]$ = {X ∈ $\varphi(\chi)$: $\breve{X} \subseteq X \subseteq X^{\max}$ } and ∪ means classical set union.

Hence we may interpret ψ as a union of a finite number of fuzzy intervals.

Theorem 4.6. ψ consists of only one element iff $X^{\max} = F(X^{\max})$. From the theorem given up to now we can derive the following algorithm for the construction of ψ:

1. Compute $X^{\max}$ and test whether $X^{\max}$ ∈ ψ or not. In the latter case we have ψ = ∅ , that is, inconsistency: STOP. Otherwise go to 2.

2. Construct $I^i(X^{\max})$ for i = 1, . . . , N.

3. Determine $F(X^{\max})$.

4. Compute $F_0(P^{\max})$ for instance by pairwise comparison.

Despite the principal feasibility of the algorithm the computation of $F(X^{\max})$ can be rather extensive. That is why we will give a theorem allowing a further reduction of computational effort. To this end we introduce yet the following notations:

$$F^i(X) = \{[f^i(X)] : f^i = I^i(X)\}, \tag{9}$$

$$\breve{F}(X) = \left\{ Z \in p(\chi) : Z = \bigvee_{i=1}^{N} X^i, X^i \in F_0^i(X) \right\}, \tag{10}$$

We have of course Z $\in$ F(X) iff Z = $\mathrm{V}_{i=1}^{N} X^i$ with $X^i \in F^i(X)$ ('V' denotes the max-operator being applied componentwise). Now we can state:

Theorem 4.7. The sets F(X) and $\breve{F}$ (X) have the same minimal elements for X$\in$ V, that is

$$F_0(X) = \breve{F}_0(X). \tag{11}$$

Proof. Let Z $\in F_0(X)$. Then we obtain Z$\in$F(X) from Lemma 2.1, that is, $F_0(X) \subseteq \breve{F}(X)$. Now suppose $Q \in F(X)$, i.e. $Q = V_{i=1}^{N} X^i$, $X^i \in F^i(X)$. For each X^i we find an $R^i \in F_0^i(X)$ (Lemma 2.1) such that $R^i \subseteq X^i$. But then $R = V_{i=1}^{N} R^i \in \breve{F}(X)$ and obviously R $\subseteq$ Q. Now we can apply Lemma 2.2 and get (11).

Example. Before listing up numerical data we make some useful remarks. If we take h(x, y) = min(x, y) we get

$$a\,\alpha\, b = \begin{cases} 1 & for\, a \le b, \\ b & otherwise. \end{cases} \tag{12}$$

Hence we obtain

$$\left[f^i(X)\right]_j = \begin{cases} \max\limits_{k \in j_{f^i}(j)} \underline{b}^i & for J_{f^i}(j) \neq \emptyset \\ 0 & otherwise. \end{cases} \tag{13}$$

Now we take $\chi = (\chi_1, \chi_2, \chi_3)$, $\gamma = (\eta_1, \eta_2, \eta_3)$. Further let N = 2 and

$$R_i = \begin{pmatrix} 0.1 & 0 & 0.6 \\ 0.5 & 0.1 & 0.3 \\ 0.2 & 0.9 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0.5 & 0.1 & 0.2 \\ 1.0 & 0.2 & 0.6 \\ 0.2 & 0.8 & 0.1 \end{pmatrix}.$$

Moreover we take h(x, y) = min(x, y). At first we consider

$$R_1 \circ X = B_1,\ R_2 \circ X = B_2 \tag{14}$$

With B_1 = (0.3, 0.2, 0.4) and B_2 = (0.5, 0.1, 0.3).

We notice that (14) is inconsistent: The first equation has the unique solution X* = (0.4, 0.3, 0.2) which does not fulfil the second equation. Now we provide (14) with the following tolerances:

$\underline{B_1}$ = (0.1,0.1, 0.2) and $\underline{B_2}$ = (0.3, 0, 0.1). (15)

$\overline{B_1}$ = (0.5, 0.3, 0.6) and $\overline{B}_2$ = (0.7, 0.2, 0.5).

Applying Theorem 4.2 we obtain $\psi \neq \emptyset$ with $X^{\max}$ = (1.0, 0.5, 0.2). Therefore we get

$I_1^1(X^{\max})$ = {1, 2, 3), I_2^1 = {2, 3}, I_3^1= {1, 2},

I_1^2= {1, 2), I_2^2 ={1, 2, 3, }, I_3^2 = 1, 2, 3).

Omitting elementary computations we get

F_0^1 = {(0.2, 0.1, 0), (0.1, 0, 0. l), (0, 0.2, 0)},

F_0^2= {(0.3, 0, 0), (0, 0.3, 0)}

which leads to $\psi = F_0 = \breve{F}_0$={(0.3, 0.1, 0), (0.3, 0, 0.1), (0, 0.3, 0)}={$\breve{X}_1, \breve{X}_2, \breve{X}_3$} and therefore

$\psi = \bigcup_{i=1}^{3} \left[\breve{X}_1, X^{\max}\right]$ using Theorem 4.5. For the case $\underline{B}_1 = B_1 = \overline{B}_1$, ($\underline{B_2}$,$\overline{B_2}$ as before) we get

$X^{\max}$ = (0.4, 0.3, 0.2) and of course $X^{\max}$ =X*. When formally using our algorithm we obtain

I_1^1 = {2}, I_2^1 = {3}, I_3^1= {l}, I_1^2=(l), I_2^2={1,2,3}, I_3^2={1,2,3} and therefore F_0^1 = {(0.4, 0.3, 0.2)}, F_0^2= {(0.3, 0, 0)} and hence it follows that $\breve{F}$ = {(0.4, 0.3, 0.2)}. But then $\psi = \breve{F}_0 = X^{\max}$, that is, $\psi = X^{\max}$ as already stated above.

5. The inverse problems (P_I^1)) and (P_I^2)

5.1. The problem (P_I^1)

Before investigating the structure of the solution set of the problem (the formulation of which has been given in the introduction) we again shall give necessary notations. So we define

$$\varphi^N(\gamma) = \{(y_1,...,y_N) : Y_j \in \varphi(\gamma); j = 1,...,N\} \quad (16)$$

We assume this set to be a poset by componentwise ordering: With $Y^1, Y^2 \in \varphi^N(\gamma)$ we define $Y^1 \le Y^2$ iff $Y_j^1 \subseteq Y_j^2$, j=l,..., N, when using the notation $Y^i = (Y_1^i,...,Y_N^i)$, i = 1,2. Now we will define the set of solutions of (P_I^1):

$$\beta(\overline{B}, B) = \{\underline{B} \in \varphi^N(\gamma) : \text{(1) is consistent for given}$$
$$B, \overline{B} \in \varphi^N(\gamma) \text{ with } \underline{B} \le B \le \overline{B}\}. \quad (17)$$

We have of course $\underline{\beta} \subseteq \varphi^N(\gamma)$. The following statement gives us the possibility to determine the extremal elements of β:

Theorem 5.1. The set β has a greatest element $\underline{B}^{\max}$ which can be computed in the following way:

$$\underline{B}_1^{\max} = B_i \Lambda R_i \circ X^{\max}, i = 1,...N, \quad (18)$$

where $X^{\max}$ is the fuzzy set given by Theorem 4.1.

Proof. First we show $\underline{B}^{\max} \in \beta$. From (18) we get immediately, $\underline{B}_i^{\max} \subseteq B_i, \underline{B}_i^{\max} \subseteq R_i \circ X^{\max}$.

Moreover $R_i \circ X^{\max} \subseteq \overline{B}_i$. But then $\underline{B}_i^{\max} \subseteq R_i \circ X^{\max} \subseteq \overline{B}_i$, i= N. Hence (1) has at least the solution $X^{\max}$. Obviously we have $\underline{B} \in \beta$ if $\underline{B} \le \underline{B}^{\max}$. Now we assume $\underline{B}^* \in \beta$, but $\underline{B}^* \not\le \underline{B}^{\max}$. Then there is an i* $\in \{1, \ldots N\}$ with $\underline{B}_{i^\bullet}^* \subseteq \underline{B}_{i^\bullet}^{\max}$ Hence $\underline{B}_{i^\bullet} \not\subseteq R_{i^\bullet} \circ X^{\max} (*)$. On the other hand there is a Z* $\in \varphi(\chi)$ with $\underline{B}_{i^\bullet} \subseteq R_{i\bullet} \circ Z^* \subseteq \overline{B}_{i\bullet}$ and from Theorem 4.1 we have Z* $\subseteq X^{\max}$ and $\underline{B}_{i^*}^* \subseteq R_{i^\bullet} \circ Z^* \subseteq R_{i\bullet} \circ X^{\max} (**)$.

But (*) and (**) form a contradiction, which ends the proof.

We will end this section with the remark that β always possesses a least element, namely the empty fuzzy set.

5.2. The problem (P_I^2)

Now we deal with the characterization of the set of feasible upper tolerances when the lower ones are given. Again we start with definitions: For a fuzzy set under consideration we denote the maximal one (over the corresponding basic space) by E. So $E \in \varphi^N(\gamma)$ means $\mu_{Ei}(\eta_j)$ = 1, i = 1, . . . , N; j = 1, . . . , n, etc.

Now we define the counterpart of β from the last section:

$$\beta\ (\underline{B}, B) = \{\overline{B} \in \varphi^N(\gamma) : \text{(1) consistent for given}$$
$$\underline{B}, B \in \varphi^N(\gamma) \text{ with } \underline{B} \le B \le \overline{B}\} \quad (19)$$

While in the previous section we could state that β is never empty it may happen that β is. Namely, let us choose $\underline{B} = E \in \varphi^N(\gamma)$. Then we compute the corresponding $\underline{B}^{\max}$ according to the above theory. It is clear that for $\underline{B} \not\ge \underline{B}^{\max}$ the set β will be empty. Nevertheless, we have the following simple proposition: If $\beta \ne 0$ then $\overline{B}^{\max} = E$ (as defined above) is the greatest element (and vice versa).

Now we will take up the determination of minimal elements of β, that is β. We consider the following problem:

Find $X \in \varphi(\chi)$ with

$$\underline{B}_i \subseteq R_i \circ X \subseteq \overline{B}_i^{\max} (= E), \text{ i=l,. . . ,N.} \quad (20)$$

We denote the solution set of this problem by ψ'. According to Theorem 4.2 we know that $\psi' \ne 0$ iff $X'^{\max} \in \psi'$ with $X'^{\max} = E \in \varphi(\chi)$. From Theorem 4.4 we have $\psi'_0 = F_0(E)$. Let

$$\beta = \{\overline{B} \in \varphi^N(\gamma) : \overline{B}_i = R_i \circ \breve{X} \vee B_i, \breve{X} \in \psi'_0, \text{ i=l,...,N}\}. \quad (21)$$

Then we can state:

Theorem 5.2. The sets $\widetilde{\beta}$ and $\overline{\beta}$ have the same minimal elements, i.e. $\widetilde{\beta}_0 = \overline{\beta_0}$.

Proof. First we show $\widetilde{\beta}_0 \subseteq \overline{\beta_0}$. To this end take $\overline{B}^* \in \widetilde{\beta}_0$, i.e. $\overline{B}_i^* = R_i \circ \breve{X}^* \vee B_i$, i=l,..., N, with the supposition $\breve{X}^* \in \psi'_0$. Now let $\overline{B} \in \overline{\beta}$ and $\overline{B} \le \overline{B}^*$. On the other hand we have $Z \in \varphi(\chi)$ with $\underline{B}_i \subseteq R_i \circ Z \subseteq \overline{B}_i$, i = 1, . . . , N, and of course $Z \in \psi'_0$. But then there exists an $\breve{X}_z \in \psi'_0$ with $\breve{X}_z \subseteq Z$ an fulfilling

$$\overline{B}_i \subseteq R_i \circ \breve{X}_z \subseteq R_i \circ Z \subseteq \overline{B}_i .$$

With the notations $\overline{B}_i^z = R_i \circ \breve{X}_z \vee B_i$ and $\overline{B}^z = (B_1^z, ..., B_N^z)$ we obtain $\overline{B}^z \le \overline{B} \le \overline{B}^*$. Since $\overline{B}^z \in \widetilde{\beta}$ we get $\overline{B}^z = \overline{B} = \overline{B}^*$. But this means $\overline{B}^* \in \overline{\beta_0}$, i.e. $\widetilde{\beta}_0 \in \overline{\beta_0}$. In a simlar way we can show that $\widetilde{\beta}_0 \subset \overline{\beta_0}$, go (strong inclusion) cannot hold.

Example. First we consider (P_I^1) with the data of the above example, where $\overline{B}_1$ and $\overline{B}_2$ are taken from (15). We will determine the maximal lower tolerances according to (18). We have already obtained that $X^{\max}$ = (1.0, 0.5, 0.2). Further, $R_1 \circ X^{\max}$=(0.5, 0.2, 0.6), $R_2 \circ X^{\max}$=(0.5, 0.2, 0.5). Hence we get $\underline{B}_1^{\max}$ = (0.3, 0.2, 0.4), $\underline{B}_2^{\max}$ = (0.5, 0.1, 0.3). This shows that the lower tolerances from (15) are not 'optimal' in the sense of feasibility.

For illustration of (P_I^2) we choose the lower tolerances from (15) and determine the set $\overline{\beta_0}$, due to Theorem 5.2. After corresponding computations we obtain $\psi'_0 = \psi_0$ with ψ_0 as in Section 4. From this we have $\overline{\beta_0}$ = (B_1, B_2), that means, the right-hand sides of (14) are the minimal upper tolerances in this example.

6. Concluding remarks

In the present paper we investigated the structure of solutions of a certain class of fuzzy equation systems. It turns out that the solution set can be described in the terms of poset theory. The applied method can be extended to problems of fuzzy modelling with tolerances (i.e., we look for a fuzzy relation when fuzzy inputs and outputs are given). This will be matter of further investigation. Besides we have interesting connections to fuzzy eigen set problems in given regions.

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Поступила в редакцию 15.02.2007

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